# The superior Method of Summing further promoted *

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**§167** To cure the defect of the method of summation we treated before, in this chapter we will consider series of such a kind, whose general terms are more complex. Since the expression found before in the geometric progressions, even though by means of other methods they can most easily be summed, does not yield the true sum contained in a finite formula, series of such a kind, whose terms are products of terms of a geometric series and an arbitrary other one, will be contemplated here at first. Therefore, let this series be propounded

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & & x \\ s = & ap + & bp^2 + & cp^3 + & dp^4 + & \cdots\cdots & + yp^x, \end{array}$$

which is composed of the geometric series $p$, $p^2$, $p^3$ etc. and another arbitrary series $a + b + c + d +$ etc., whose general term or the one corresponding to index $x$ is $= y$, and let us investigate the general term for the value of its sum $s = S.yp^x$.

**§168** Let us perform the calculation the same way as we did above, and let $v$ be the term preceding $y$ in the series $a + b + c + d +$ etc. and $A$ the one preceding $a$ or the one corresponding to the index $0$, $vp^{x-1}$ will be the general term of this series

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$$\begin{array}{llllll} 1 & 2 & 3 & 4 & & x \\ A & = ap + bp^2 + cp^3 + & \cdots\cdots & + yp^x; \end{array}$$

if its sum is indicated by $S.vp^{x-1}$, it will be

$$S.vp^{x-1} = \frac{1}{p}S.vp^x = S.yp^x - yp^x + A.$$

But because it is

$$v = y - \frac{dy}{dx} + \frac{ddy}{2dx^2} - \frac{d^3y}{6dx^3} + \frac{d^4y}{24dx^4} - \frac{d^5y}{120dx^5} + \text{etc.},$$

it will be

$$S.yp^x - yp^x + A = \frac{1}{p}S.yp^x - \frac{1}{p}S.\frac{dy}{dx}p^x + \frac{1}{2p}S.\frac{ddy}{dx^2}p^x$$
$$-\frac{1}{6p}S.\frac{d^3y}{dx^3}p^x + \frac{1}{24p}S.\frac{d^4y}{dx^4}p^x - \text{etc.}$$

From this it is

$$S.yp^x = \frac{1}{p-1}\left(yp^{x+1} - Ap - S.\frac{dy}{dx}p^x + S.\frac{ddy}{2dx^2}p^x - S.\frac{d^3y}{6dx^3} + \text{etc.}\right)$$

If one therefore has the summatory terms of the series whose general terms are $\frac{dy}{dx}p^x$, $\frac{ddy}{dx^2}p^x$, $\frac{d^3y}{dx^3}p^x$ etc., from them one will be able to define the summatory term $S.yp^x$.

**§169** Hence, one will be able to find the sums of the series whose general terms are contained in this form $x^n p^x$. For, let $y = x^n$; it will be $A = 0$, if it is not $n = 0$, in which case $A = 1$, and since it is

$$\frac{dy}{dx} = nx^{n-1}, \quad \frac{ddy}{2dx^2} = \frac{n(n-1)}{1\cdot 2}x^{n-2}, \quad \frac{d^3y}{6dx^3} = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3} \quad \text{etc.},$$

it will be

$$S.x^n p^x = \frac{1}{p-1}\left\{\begin{array}{c} x^n p^{x+1} - Ap - nS.x^{n-1}p^x + \frac{n(n-1)}{1\cdot 2}S.x^{n-2}p^x \\ -\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}S.x^{n-3}p^x + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}S.x^{n-4}p^x - \text{etc.}\end{array}\right\}$$

From this form now by successively substituting the numbers 0, 1, 2, 3 etc. for $n$ one will obtain the following summations; and at first, if $n = 0$, it is $A = 1$, but in the remaining cases it will be $A = 0$.

$$S.x^0 p^x = S.p^x = \frac{1}{p-1}(p^{x+1} - p) = \frac{p^{x+1} - p}{p-1} = \frac{p(p^x - 1)}{p-1},$$

which is the known sum of the geometric progression;

$$S.xp^x = \frac{1}{p-1}(xp^{x+1} - S.p^x) = \frac{xp^{x+1}}{p-1} - \frac{p^{x+1} - p}{(p-1)^2}$$

or

$$S.xp^x = \frac{pxp^x}{p-1} - \frac{p(p^x - 1)}{(p-1)^2};$$

$$S.x^2 p^x = \frac{1}{p-1}(x^2 p^{x+1} - 2S.xp^x + S.p^x)$$

or

$$S.x^2 p^x = \frac{x^2 p^{x+1}}{p-1} - \frac{2xp^{x+1}}{(p-1)^2} + \frac{p(p+1)(p^x - 1)}{(p-1)^3}.$$

Further, it is

$$S.x^3 p^x = \frac{1}{p-1}(x^3 p^{x+1} - 3S.x^2 p^x + 3S.xp^x - S.p^x)$$

or

$$S.x^3 p^x = \frac{x^3 p^{x+1}}{p-1} - \frac{3x^2 p^{x+1}}{(p-1)^2} + \frac{3(p+1)xp^{x+1}}{(p-1)^3} - \frac{p(pp + 4p + 1)(p^x - 1)}{(p-1)^4}$$

and proceeding further this way one will be able to define the sums of the superior powers $x^4 p^x$, $x^5 p^x$, $x^6 p^x$ etc.; but this is done in a more convenient way by means of the general expression we will now investigate.

**§170** Since we found that it is

$$S.yp^x = \frac{1}{p-1}\left(yp^{x+1} - Ap - S.\frac{dy}{dx}p^x + S.\frac{ddy}{2dx^2}p^x - S.\frac{d^3y}{6dx^3}p^x + \text{etc.}\right),$$

where $A$ is a constant of such a kind that the sum becomes $= 0$, if one puts $x = 0$ (for, in this case it is $y = A$ and $yp^{x+1} = Ap$), we will be able to omit this constant, if we only continuously remember that to a certain sum always a constant of such a kind is to be added that having put $x = 0$ it vanishes or that another certain case is satisfied. Therefore, instead of $y$ let us put $z$ and it will be

$$S.p^x z = \frac{p^{x+1}z}{p-1} - \frac{1}{p-1}S.p^x\frac{dz}{dx} + \frac{1}{2(p-1)}S.p^x\frac{ddz}{dx^2} - \frac{1}{6(p-1)}S.p^x\frac{d^3z}{dx^3}$$
$$+\frac{1}{24(p-1)}S.p^x\frac{d^4z}{dx^4} - \frac{1}{120(p-1)}S.p^x\frac{d^5z}{dx^5} + \text{etc.}$$

Furthermore, let us successively put $\frac{dz}{dx}$, $\frac{ddz}{dx^2}$, $\frac{d^3z}{dx^3}$ etc. instead of $y$ and it will be

$$\begin{aligned}
S.\frac{p^x dz}{dx} &= \frac{p^{x+1}}{p-1}\cdot\frac{dz}{dx} - \frac{1}{p-1}S.\frac{p^x ddz}{dx^2} + \frac{1}{2(p-1)}S.\frac{p^x d^3z}{dx^3} - \text{etc.}\\
S.\frac{p^x ddz}{dx^2} &= \frac{p^{x+1}}{p-1}\cdot\frac{ddz}{dx^2} - \frac{1}{p-1}S.\frac{p^x d^3z}{dx^3} + \frac{1}{2(p-1)}S.\frac{p^x d^4z}{dx^4} - \text{etc.}\\
S.\frac{p^x d^3z}{dx^3} &= \frac{p^{x+1}}{p-1}\cdot\frac{d^3z}{dx^3} - \frac{1}{p-1}S.\frac{p^x d^4z}{dx^4} + \frac{1}{2(p-1)}S.\frac{p^x d^5z}{dx^5} - \text{etc.}\\
&\text{etc.}
\end{aligned}$$

Therefore, if these values are successively substituted, $S.p^x z$ will be expressed by a form of this kind

$$S.p^x z = \frac{p^{x+1}z}{p-1} - \frac{\alpha p^{x+1}}{p-1}\cdot\frac{dz}{dx} + \frac{\beta p^{x+1}}{p-1}\cdot\frac{ddz}{dx^2} - \frac{\gamma p^{x+1}}{p-1}\cdot\frac{d^3z}{dx^3}$$
$$+\frac{\delta p^{x+1}}{p-1}\cdot\frac{d^4z}{dx^4} - \frac{\varepsilon p^{x+1}}{p-1}\cdot\frac{d^5z}{dx^5} + \text{etc.}$$

**§171** To define the values of the letters $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$ etc. for each term substitute the series found before, namely,

$$\frac{p^{x+1}z}{p-1} = S.p^x z + \frac{1}{p-1}S.\frac{p^x dz}{dx} - \frac{1}{2(p-1)}S.\frac{p^x ddz}{dx^2} + \frac{1}{6(p-1)}S.\frac{p^x d^3z}{dx^3} - \text{etc.}$$
$$\frac{p^{x+1}dz}{(p-1)dx} = S.\frac{p^x dz}{dx} + \frac{1}{p-1}S.\frac{p^x ddz}{dx^2} - \frac{1}{2(p-1)}S.\frac{p^x d^3z}{dx^3} + \text{etc.}$$
$$\frac{p^{x+1}ddz}{(p-1)dx^2} = S.\frac{p^x ddz}{dx^2} + \frac{1}{p-1}S.\frac{p^x d^3z}{dx^3} - \text{etc.}$$
$$\frac{p^{x+1}d^3z}{(p-1)dx^3} = S.\frac{p^x d^3z}{dx^3} + \text{etc.}$$

Therefore, we will have

$$S.p^x z = S.p^x z$$

$$\begin{array}{llll}
+\dfrac{1}{p-1}S.\dfrac{p^x dz}{dx} & -\dfrac{1}{2(p-1)}S.\dfrac{p^x ddz}{dx^2} & +\dfrac{1}{6(p-1)}S.\dfrac{p^x d^3z}{dx^3} & -\dfrac{1}{24(p-1)}S.\dfrac{p^x d^4z}{dx^4} + \text{etc.} \\
-\alpha & -\dfrac{\alpha}{p-1} & +\dfrac{\alpha}{2(p-1)} & -\dfrac{\alpha}{6(p-1)} \\
 & +\beta & +\dfrac{\beta}{p-1} & -\dfrac{\beta}{2(p-1)} \\
 & & -\gamma & -\dfrac{\gamma}{p-1} \\
 & & & +\delta
\end{array}$$

whence the following values of the coefficients $\alpha$, $\beta$, $\gamma$, $\delta$ etc. will be obtained

$$\alpha = \frac{1}{p-1}, \quad \beta = \frac{1}{p-1}\left(\alpha + \frac{1}{2}\right), \quad \gamma = \frac{1}{p-1}\left(\beta + \frac{\alpha}{2} + \frac{1}{6}\right),$$
$$\delta = \frac{1}{p-1}\left(\gamma + \frac{\beta}{2} + \frac{\alpha}{6} + \frac{1}{24}\right), \quad \varepsilon = \frac{1}{p-1}\left(\delta + \frac{\gamma}{2} + \frac{\beta}{6} + \frac{\alpha}{24} + \frac{1}{120}\right)$$
$$\text{etc.}$$

**§172** For the sake of brevity, let $\frac{1}{p-1} = q$; it will be

$$\begin{aligned}
\alpha &= q \\
\beta &= \alpha q + \frac{1}{2}q = qq + \frac{1}{2}q \\
\gamma &= \beta + \frac{1}{2}\alpha q + \frac{1}{6}q = q^3 + qq + \frac{1}{6}q \\
\delta &= \gamma q + \frac{1}{2}\beta q + \frac{1}{6}\alpha q + \frac{1}{24}q = q^4 + \frac{3}{2}q^3 + \frac{7}{12}q^2 + \frac{1}{24}q \\
\varepsilon &= \delta q + \frac{1}{2}\gamma q + \frac{1}{6}\beta q + \frac{1}{24}\alpha q + \frac{1}{120}q = q^5 + 2q^4 + \frac{5}{4}q^3 + \frac{1}{4}q^2 + \frac{1}{120}q \\
\zeta &= q^6 + \frac{5}{2}q^5 + \frac{13}{6}q^4 + \frac{3}{4}q^3 + \frac{31}{360}q^2 + \frac{1}{720}q \\
&\text{etc.}
\end{aligned}$$

or express them this way

$$\begin{aligned}
\alpha &= \frac{q}{1} \\
\beta &= \frac{2qq + q}{1 \cdot 2} \\
\gamma &= \frac{6q^3 + 6q^2 + q}{1 \cdot 2 \cdot 3} \\
\delta &= \frac{24q^4 + 36q^3 + 14q^2 + q}{1 \cdot 2 \cdot 3 \cdot 4} \\
\varepsilon &= \frac{120q^5 + 240q^4 + 150q^3 + 30q^2 + q}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \\
\zeta &= \frac{720q^6 + 1800q^5 + 1560q^4 + 540q^3 + 62q^2 + q}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \\
\eta &= \frac{5040q^7 + 15120q^6 + 16800q^5 + 8400q^4 + 1806q^3 + 126q^2 + q}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \\
&\text{etc.,}
\end{aligned}$$

where the coefficient 16800 arises, if the sum of the two superior $1560 + 1800$ is multiplied by the exponent of $q$, which is 5 here.

**§173** Let us resubstitute the value $\frac{1}{p-1}$ instead of $q$

$$\alpha = \frac{1}{1(p-1)}$$

$$\beta = \frac{p+1}{1\cdot 2(p-1)^2}$$

$$\gamma = \frac{pp+4p+1}{1\cdot 2\cdot 3(p-1)^3}$$

$$\delta = \frac{p^3+11p^2+11p+1}{1\cdot 2\cdot 3\cdot 4(p-1)^4}$$

$$\varepsilon = \frac{p^4+26p^3+66p^2+26p^2+1}{1\cdot 2\cdot 3\cdot 4\cdot 5(p-1)^5}$$

$$\zeta = \frac{p^5+57p^4+302p^3+302p^2+57p+1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6(p-1)^6}$$

$$\eta = \frac{p^6+120p^5+1191p^4+2416p^3+1191p^2+120p+1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7(p-1)^7}$$

etc.

The law of these quantities behaves as this that, if any term is set

$$= \frac{p^{n-2}+Ap^{n-3}+Bp^{n-4}+Cp^{n-5}+Dp^{n-6}+\text{etc.}}{1\cdot 2\cdot 3\cdots(n-1)(p-1)^{n-1}},$$

it will be

$$A = 2^{n-1}-n$$

$$B = 3^{n-1}-n\cdot 2^{n-1}+\frac{n(n-1)}{1\cdot 2}$$

$$C = 4^{n-1}-n\cdot 3^{n-1}+\frac{n(n-1)}{1\cdot 2}2^{n-1}-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}$$

$$D = 5^{n-1}-n\cdot 4^{n-1}+\frac{n(n-1)}{1\cdot 2}3^{n-1}-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}2^{n-1}+\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}$$

etc.,

whence these coefficients $\alpha$, $\beta$, $\gamma$, $\delta$ etc. can be continued as far as one desires.

**§174** But if we on the other hand consider the law, according to which these coefficients depend on each other, it will easily become clear that they constitute a recurring series and arise, if this fraction is expanded

$$\frac{1}{1-\frac{u}{p-1}-\frac{u^2}{2(p-1)}-\frac{u^3}{6(p-1)}-\frac{u^4}{24(p-1)}-\text{etc.}};$$

for, this series will arise

$$1+\alpha u+\beta u^2+\gamma u^3+\delta u^4+\varepsilon u^5+\zeta u^6+\text{etc.};$$

Put that fraction $= V$, and because it is

$$V=\frac{p-1}{p-1-u-\frac{u^2}{2}-\frac{u^3}{6}-\frac{u^4}{24}-\text{etc.}},$$

it will be

$$V=\frac{p-1}{p-e^u},$$

where $e$ is the number whose hyperbolic logarithm is $= 1$. And if the value of $V$ is expressed by means of a power series in $u$, it will arise

$$V=1+\alpha u+\beta u^2+\gamma u^3+\delta u^4+\varepsilon u^5+\zeta u^6+\text{etc.},$$

whose coefficients $\alpha$, $\beta$, $\gamma$, $\delta$ etc. will be those, we need in the present task. Therefore, having found those, it will be

$$S.p^x z=\frac{p^{x+1}}{p-1}\left(z-\frac{\alpha dz}{dx}+\frac{\beta ddz}{dx^2}-\frac{\gamma d^3z}{dx^3}+\frac{\delta d^4z}{dx^4}-\text{etc.}\right)\pm\text{Const.},$$

which expression therefore is the summatory of this series

$$ap+bp^2+cp^3+\cdots+p^x z,$$

whose general term is $= p^x z$.

**§175** Since we found that it is $V=\frac{p-1}{p-e^u}$, it will be

$$e^u=\frac{pV-p+1}{V}$$

and by taking logarithms it will be

$$u=\ln(pV-p+1)-\ln V$$

and hence by differentiating

$$du = \frac{(p-1)dV}{pV^2 - (p-1)V},$$

whence it will be

$$pV^2 = (p-1)V + \frac{(p-1)dV}{du}.$$

Since therefore it is

$$V = 1 + \alpha u + \beta u^2 + \gamma u^3 + \delta u^4 + \varepsilon u^5 + \text{etc.},$$

it will be

$$\begin{aligned} pV^2 = p \ + \ 2\alpha u + 2\beta pu^2 &+ 2\gamma pu^3 \ + 2\delta pu^4 \ + 2\varepsilon pu^5 \ + \text{etc.} \\ &+ \alpha^2 pu^2 \ + 2\alpha\beta pu^3 + 2\alpha\gamma pu^4 + 2\alpha\delta pu^5 \ + \text{etc.} \\ &+ \beta\beta pu^4 \ + 2\beta\gamma pu^5 + \text{etc.} \end{aligned}$$

$$\begin{aligned} (p-1)V &= (p-1) + \alpha(p-1)u + \beta(p-1)u^2 + \gamma(p-1)u^3 \\ &\quad + \delta(p-1)u^4 + \varepsilon(p-1)u^5 + \text{etc.} \\ \frac{(p-1)dV}{du} &= (p-1)\alpha + 2(p-1)\beta u + 2(p-1)\gamma u^2 + 4(p-1)\delta u^3 \\ &\quad + 5(p-1)\varepsilon u^4 + 6(p-1)\zeta u^5 + \text{etc.}, \end{aligned}$$

equating which expressions one will find

$$\begin{aligned} 1(p-1)\alpha &= \ 1 \\ 2(p-1)\beta &= \alpha(p+1) \\ 3(p-1)\gamma &= \beta(p+1) + \alpha^2 p \\ 4(p-1)\delta &= \gamma(p+1) + 2\alpha\beta p \\ 5(p-1)\varepsilon &= \delta(p+1) + 2\alpha\gamma p + \beta\beta p \\ 6(p-1)\zeta &= \varepsilon(p+1) + 2\alpha\delta p + 2\beta\gamma p \\ 7(p-1)\eta &= \zeta(p+1) + 2\alpha\varepsilon p + 2\beta\delta p + \gamma\gamma p \\ &\text{etc.} \end{aligned}$$

from which formulas, if for $p$ a given number is assumed, the values of the coefficients $\alpha$, $\beta$, $\gamma$, $\delta$ etc. can be determined more easily than from the law first found.

**§176** Before we descend to special cases concerning the value $p$, let us put that it is $z = x^n$ such that this series has to be summed

$$s = p + 2^n p^2 + 3^n p^3 + 4^n p^4 + \cdots + x^n p^x,$$

and it will be by means of the expression found before

$$s = p^x \left\{ \begin{array}{c} \dfrac{p}{p-1} \cdot x^n - \dfrac{p}{(p-1)^2} n x^{n-1} + \dfrac{pp+p}{(p-1)^3} \cdot \dfrac{n(n-1)}{1 \cdot 2} x^{n-2} \\ - \dfrac{p^3+4p^2+p}{(p-1)^4} \cdot \dfrac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^{n-3} + \text{etc.} \end{array} \right\}$$

$\pm C$, which makes $s = 0$, if one puts $x = 0$.

Hence, by successively putting the numbers 0, 1, 2, 3, 4 etc. for $n$ it will be

$$S.x^0 p^x = p^x \frac{p}{p-1} - \frac{p}{p-1}$$

$$S.x^1 p^x = p^x \left( \frac{px}{p-1} - \frac{p}{(p-1)^2} \right) + \frac{p}{(p-1)^2}$$

$$S.x^2 p^x = p^x \left( \frac{px^2}{p-1} - \frac{2px}{(p-1)^2} + \frac{p(p+1)}{(p-1)^3} \right) - \frac{p(p+1)}{(p-1)^3}$$

$$S.x^3 p^x = p^x \left( \frac{px^3}{p-1} - \frac{3px^2}{(p-1)^2} + \frac{3p(p+1)x}{(p-1)^3} - \frac{p(p^2+4p+1)}{(p-1)^4} \right) + \frac{p(p^2+4p+1)}{(p-1)^4}$$

$$S.x^4 p^x = p^x \left( \frac{px^4}{p-1} - \frac{4px^3}{(p-1)^2} + \frac{6p(p+1)x^2}{(p-1)^3} - \frac{4p(p^2+4p+1)x}{(p-1)^4} + \frac{p(p^3+11p^2+11p+1)}{(p-1)^5} \right) - \frac{p(p^3+11p^2+11p+1}{(p-1)^5}$$

$$S.x^5 p^x = \frac{p^{x+1}x^5}{p-1} - \frac{5p^{x+1}x^4}{(p-1)^2} + \frac{10(p+1)p^{x+1}x^3}{(p-1)^3} - \frac{10(p^2+4p+1)p^{x+1}x^2}{(p-1)^4} + \frac{5(p^3+11p^2+11p+1)p^{x+1}x}{(p-1)^5} - \frac{(p^4+26p^3+66p^2+26p+1)(p^{x+1}-p)}{(p-1)^6}$$

$$S.x^6p^x = \frac{p^{x+1}x^6}{p-1} - \frac{6p^{x+1}x^5}{(p-1)^2} + \frac{15(p+1)p^{x+1}x^4}{(p-1)^3} - \frac{20(p^2+4p+1)(p^{x+1}-p)}{(p-1)^4}$$
$$+\frac{15(p^3+11p^2+11p+1)p^{x+1}x^2}{(p-1)^5} - \frac{6(p^4+26p^3+66p^2+26p+1)p^{x+1}x}{(p-1)^6}$$
$$+\frac{(p^5+57p^4+302p^3+302p^2+57p+1)(p^{x+1}-p)}{(p-1)}$$
etc.

**§177** Hence, it is understood, if $z$ was a polynomial function of $x$, that then the sum of the series whose general term is $p^xz$ can be exhibited, since by taking the differentials of $z$ one finally gets to vanishing ones. If this series is propounded

$$p + 3p^2 + 6p^3 + 10p^4 + \cdots + \frac{xx+x}{2}p^x,$$

because of

$$z = \frac{xx+x}{2} \quad \text{and} \quad \frac{dz}{dx} = x + \frac{1}{2} \quad \text{and} \quad \frac{ddz}{dx^2} = 1$$

the summatory term will be

$$s = \frac{p^{x+1}}{p-1}\left(\frac{1}{2}xx + \frac{1}{2}x - \frac{2x+1}{2(p-1)} + \frac{p+1}{2(p-1)^2}\right) - \frac{p}{p-1}\left(\frac{p+1}{2(p-1)^2} - \frac{1}{2(p-1)}\right)$$

or

$$s = p^{x+1}\left(\frac{xx}{2(p-1)} + \frac{(p-3)}{2(p-1)^2} + \frac{1}{(p-1)^3}\right) - \frac{p}{p-1}.$$

But if $z$ was not a polynomial function, then this expression of the summatory terms runs to infinity. So if $z = \frac{1}{x}$ that this series is to be summed

$$s = p + \frac{1}{2}p^2 + \frac{1}{3}p^3 + \frac{1}{4}p^4 + \cdots + \frac{1}{x}p^x,$$

because of

$$\frac{dz}{dx} = -\frac{1}{xx}, \quad \frac{ddz}{dx^2} = \frac{2}{x^3}, \quad \frac{d^3z}{dx^3} = -\frac{2\cdot 3}{x^4}, \quad \frac{d^4z}{dx^4} = \frac{2\cdot 3\cdot 4}{x^5} \quad \text{etc.}$$

this summatory term will arise

$$s = \frac{p^{x+1}}{p-1}\left(\frac{1}{x} + \frac{1}{(p-1)x^2} + \frac{p+1}{(p-1)^2x^3} + \frac{pp+4p+1}{(p-1)^3x^4} + \frac{p^3+11p^2+11p+1}{(p-1)^4x^5} + \text{etc.}\right) + C.$$

In this case the constant $C$ can therefore not be defined from the case $x = 0$; to define it, put $x = 1$, and since it is $s = p$, it will be

$$C = p - \frac{pp}{p-1}\left(1 + \frac{1}{p-1} + \frac{p+1}{(p-1)^2} + \frac{pp+4p+1}{(p-1)^3} + \text{etc.}\right).$$

**§178** From these things it is perspicuous, if $p$ does not denote a determined number, that hardly anything of use to exhibit the sums of series approximately follows from this. But at first it is plain that for $p$ one cannot write 1, since all coefficients $\alpha$, $\beta$, $\gamma$, $\delta$ etc. would become infinitely large. Hence, because the series, we treat now, goes over into that one we already contemplated before, if one puts $p = 1$, it is wondrous, that this case cannot be found as the most simple case from this. Then, on the other hand it is also notable that in the case $p = 1$ the summation requires the integral $\int zdx$, although in general the sum can be exhibited without a single integral. So it happens that, while all the coefficients $\alpha$, $\beta$, $\gamma$, $\delta$ etc. grow to infinity, at the same time that integral formula is brought in. And this case, in which $p = 1$, is the only one to which this general expression found here cannot be applied. But even in this case the general formula is not to be considered to recede from the true value; for, even though the single terms become infinite, nevertheless all the infinities indeed cancel each other and a finite quantity remains equal and congruent to that one we found by means of the first method what we will elaborate in more detail below.

**§179** Therefore, let $p = -1$ and the signs in the series to be summed will alternate

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & & x \\ -\,a & +\,b & -\,c & +\,d & -\cdots & \pm\,z, \end{array}$$

where $z$ will be affirmative, if $x$ was an even number, but negative, if $x$ is an odd number. Therefore, having put

$$-a + b - c + d - \cdots \pm z = s,$$

it will be

$$s = \pm\frac{1}{2}\left(z - \frac{\alpha dz}{dx} + \frac{\beta ddz}{dx^2} - \frac{\gamma d^3z}{dx^3} + \frac{\delta d^4z}{dx^4} - \text{etc.}\right) + C,$$

where the superior of the ambiguous signs holds, if $x$ us an even number, the other, if $x$ is an odd number. Therefore, by changing the sings, it will be

$$a - b + c - d + e - f + \cdots \mp z = \mp\frac{1}{2}\left(z - \frac{\alpha dz}{dx} + \frac{\beta ddz}{dx^2} - \frac{\gamma d^3z}{dx^3} + \frac{\delta d^4z}{dx^4} - \text{etc.}\right) + C,$$

where the ambiguity of the signs follows the same law.

**§180** In this case the coefficients $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$, $\zeta$ etc. can be found from the values given before by putting $p = -1$ everywhere. But they will be found more easily from the general formulas given in § 175 from which it is at the same time understood that each second of these coefficients vanish. For, having put $p = -1$ these formulas will go over into

$$-2\alpha = 0, \quad -4\beta = 0, \quad -6\gamma = 0 - \alpha^2, \quad -8\delta = 0 - 2\alpha\beta,$$
$$-10\varepsilon = 0 - 2\alpha\gamma - \beta\beta, \quad -12\zeta = 0 - 2\alpha\delta - 2\beta\gamma \quad \text{etc.}$$

whence, because $\beta = 0$, it also will be $\delta = 0$ and further $\zeta = 0$, $\theta = 0$ and the remaining letters will be determined in such a way that it is

$$\alpha = -\frac{1}{2}, \quad \gamma = \frac{\alpha^2}{6}, \quad \varepsilon = \frac{2\alpha\gamma}{10}, \quad \eta = \frac{2\alpha\varepsilon + \gamma\gamma}{14}, \quad \iota = \frac{2\alpha\eta + 2\gamma\varepsilon}{18} \quad \text{etc.}$$

**§181** That this calculus can be done in a more convenient way let us introduce new letters and let

$$\alpha = -\frac{A}{1\cdot 2}, \quad \gamma = \frac{B}{1\cdot 2\cdot 3\cdot 4}, \quad \varepsilon = -\frac{C}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6},$$
$$\eta = \frac{D}{1\cdot 2\cdot 3\cdots 8}, \quad \iota = -\frac{E}{1\cdot 2\cdot 3\cdots 10}$$
$$\text{etc.}$$

And the sum exhibited before will be

$$\mp\frac{1}{2}\left(z+\frac{Adz}{1\cdot 2dx}-\frac{Bd^3z}{1\cdot 2\cdot 3\cdot 4dx^3}+\frac{Cd^5z}{1\cdot 2\cdots 6dx^5}-\frac{Dd^7z}{1\cdot 2\cdots 8dx^7}+\text{etc.}\right)+C.$$

But the coefficients will be defined from the following formulas

$$\begin{aligned}
1A &= 1\\
3B &= \frac{4\cdot 3}{1\cdot 2}\cdot\frac{AA}{2}\\
5C &= \frac{6\cdot 5}{1\cdot 2}\cdot AB\\
7D &= \frac{8\cdot 7}{1\cdot 2}\cdot AC+\frac{8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{BB}{2}\\
9E &= \frac{10\cdot 9}{1\cdot 2}\cdot AD+\frac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4}\cdot BC\\
11F &= \frac{12\cdot 11}{1\cdot 2}\cdot AE+\frac{12\cdot 11\cdot 10\cdot 9}{1\cdot 2\cdot 3\cdot 4}\cdot BD+\frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\cdot\frac{CC}{2}\\
&\text{etc.,}
\end{aligned}$$

which can be represented easier and more accommodated to calculation in this way

$$\begin{aligned}
A &= 1\\
B &= 2\cdot\frac{AA}{2}\\
C &= 3\cdot AB\\
D &= 4\cdot AC+4\cdot\frac{6\cdot 5}{3\cdot 4}\cdot\frac{BB}{2}\\
E &= 5\cdot AC+5\cdot\frac{8\cdot 7}{3\cdot 4}\cdot BC\\
F &= 6\cdot AD+6\cdot\frac{10\cdot 9}{3\cdot 4}\cdot BD+6\cdot\frac{10\cdot 9\cdot 8\cdot 7}{\cdot 3\cdot 4\cdot 5\cdot 6}\cdot\frac{CC}{2}\\
G &= 7\cdot AE+6\cdot\frac{12\cdot 11}{3\cdot 4}\cdot BE+7\cdot\frac{12\cdot 11\cdot 10\cdot 9}{\cdot 3\cdot 4\cdot 5\cdot 6}\cdot CD\\
&\text{etc.}
\end{aligned}$$

Hence, having performed the calculation it will be found

$$\begin{aligned}
A &= 1 \\
B &= 1 \\
C &= 3 \\
D &= 17 \\
E &= 155 &&= 5 \cdot 31 \\
F &= 2073 &&= 691 \cdot 3 \\
G &= 38227 &&= 7 \cdot 5461 &&= 7 \cdot \frac{127 \cdot 129}{3} \\
H &= 929569 &&= 3617 \cdot 257 \\
I &= 28820619 &&= 43867 \cdot 9 \cdot 73 \\
& \text{etc.}
\end{aligned}$$

**§182** If we consider these numbers with more attention, from the factors 691, 3617, 43867 one can easily conclude that these numbers have a connection to the Bernoulli numbers exhibited above [§ 122] and can hence be determined. Therefore, to anyone investigating this relation it will soon become clear that these numbers can be formed from the Bernoulli numbers $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ etc. the following way.

$$\begin{aligned}
A &= 2 \cdot 1 \cdot 3\,\mathfrak{A} &&= 2(2^2 - 1)\mathfrak{A} \\
B &= 2 \cdot 3 \cdot 5\,\mathfrak{B} &&= 2(2^4 - 1)\mathfrak{B} \\
C &= 2 \cdot 7 \cdot 9\,\mathfrak{C} &&= 2(2^6 - 1)\mathfrak{C} \\
D &= 2 \cdot 15 \cdot 17\,\mathfrak{D} &&= 2(2^8 - 1)\mathfrak{D} \\
E &= 2 \cdot 31 \cdot 33\,\mathfrak{E} &&= 2(2^{10} - 1)\mathfrak{E} \\
F &= 2 \cdot 63 \cdot 67\,\mathfrak{F} &&= 2(2^{12} - 1)\mathfrak{F} \\
G &= 2 \cdot 127 \cdot 129\,\mathfrak{G} &&= 2(2^{14} - 1)\mathfrak{G} \\
H &= 2 \cdot 255 \cdot 257\,\mathfrak{H} &&= 2(2^{16} - 1)\mathfrak{H} \\
& \text{etc.}
\end{aligned}$$

Since the Bernoulli numbers are fractional, but our coefficients on the other hand are integers, it becomes plain that these factors always cancel the fractions and they will therefore be

$$\begin{aligned}
A &= 1\\
B &= 1\\
C &= 3\\
D &= 17\\
E &= 5\cdot 31 = 155\\
F &= 3\cdot 691 = 2073\\
G &= 7\cdot 43\cdot 127 = 38277\\
H &= 257\cdot 3617 = 929569\\
I &= 9\cdot 73\cdot 43867 = 28820619\\
K &= 5\cdot 31\cdot 41\cdot 174611 = 1109652905\\
L &= 89\cdot 683\cdot 854513 = 51943281713\\
M &= 3\cdot 4097\cdot 236364091 = 2905151042481\\
N &= 2731\cdot 8191\cdot 8553103 = 191329672483963\\
&\qquad\qquad \text{etc.}
\end{aligned}$$

From these numbers one will therefore vice versa be able to find the Bernoulli numbers.

**§183** Therefore, by applying these Bernoulli numbers to the propounded series

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & & x \\ a & -b & +c & -d & +e & -\cdots & \mp z \end{matrix}$$

the sum will be

$$\mp\left(\frac{1}{2}z+\frac{(2^2-1)\mathfrak{A}dz}{1\cdot 2dx}-\frac{(2^4-1)\mathfrak{B}d^3z}{1\cdot 2\cdot 3\cdot 4dx^3}+\frac{(2^6-1)\mathfrak{C}d^5z}{1\cdot 2\cdots 6dx^5}-\frac{(2^8-1)\mathfrak{D}d^7z}{1\cdot 2\cdots 8dx^7}+\text{etc.}\right)+\text{Const.}$$

But hence it is understood that these numbers do not go into this expression by accident; for, as the propounded series arises, if from this one

$$a+b+c+d+\cdots+z,$$

where all terms have the sign $+$, the sum of each second term $b+d+f+$ etc. taken twice is subtracted, so also the found expression can be resolved into two parts of which the one is the sum of all terms affected with the sign $+$ which will be

$$\int zdx + \frac{1}{2}z + \frac{\mathfrak{A}dz}{1\cdot 2dx} - \frac{\mathfrak{B}d^3z}{1\cdot 2\cdot 3\cdot 4dx^3} + \frac{\mathfrak{C}d^5z}{1\cdot 2\cdots 6dx^5} - \text{etc.}$$

But the sum of each second is found the same way in which we did it above [chap. V]. Since the last term is $z$ corresponding to the index $x$ the one corresponding to the index $x-2$ will be

$$z - \frac{2dz}{dx} + \frac{2^2ddz}{1\cdot 2dx^2} - \frac{2^3d^3z}{1\cdot 2\cdot 3dx^3} + \frac{2^4d^4z}{1\cdot 2\cdot 3\cdot 4dx^4} - \text{etc.}\,,$$

which form arises from the latter by means of which the preceding term is expressed, if instead of $x$ one writes $\frac{x}{2}$. Therefore, one will have the sum of each the second terms, if in the sum of all terms instead of $x$ one writes $\frac{x}{2}$ everywhere which will therefore be

$$\frac{1}{2}\int zdx + \frac{1}{2}z + \frac{2\mathfrak{A}dz}{1\cdot 2dx} - \frac{2^3\mathfrak{B}d^3z}{1\cdot 2\cdot 3\cdot 4dx^3} + \frac{2^5\mathfrak{C}d^5z}{1\cdot 2\cdots 6dx^5} - \text{etc.};$$

if its double is subtracted from the preceding while $x$ is an even number or if the preceding sum is subtracted from the double of this one, if $x$ is an odd number, the residue will show the sum of the series

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & & x \\ a & -\,b & +\,c & -\,d & +\,e & -\cdots & \mp z, \end{matrix}$$

which will therefore be

$$\mp\left(\frac{1}{2}z + \frac{(2^2-1)\mathfrak{A}dz}{1\cdot 2dx} - \frac{(2^4-1)\mathfrak{B}d^3z}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}\right) + C,$$

which is the same expression we just found.

**§184** For $z$ take a power of $x$, namely $x^n$, that one finds the sum of the series

$$1 - 2^n + 3^n - 4^n + \cdots - \pm x^n$$

Because of

$$\frac{dz}{1dx} = \frac{n}{1}x^{n-1}, \quad \frac{d^3z}{1\cdot 2\cdot 3dx^3} = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3} \quad \text{etc.}$$

by using the coefficients $A$, $B$, $C$, $D$, $E$ etc. it will be

$$\mp\frac{1}{2}\left\{\begin{array}{l} x^n + \frac{A}{2}nx^{n-1} - \frac{B}{4}\cdot\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3} + \frac{C}{6}\cdot\frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^{n-5} \\ \qquad - \frac{D}{8}\cdot\frac{n(n-1)\cdots(n-6)}{1\cdot 2\cdots 7}x^{n-7} + \text{etc.} + \text{Const.} \end{array}\right\}$$

where the superior sign holds, if $x$ is an even number, the inferior on the other hand, if it is an odd number. But the constant has to defined in such a way that the sum vanishes, if $x = 0$, in which case the superior sign holds. By successively substituting the numbers 0, 1, 2, 3 etc. instead of $n$ the following sums will arise

$$\text{I.}\quad 1 - 1 + 1 - 1 + \cdots \mp 1 = \mp\frac{1}{2}(1) + \frac{1}{2};$$

if the number of terms was even, the sum will be $= 0$, if odd, it will be $= +1$.

$$\text{II.}\quad 1 - 2 + 3 - 4 + \cdots \mp x = \mp\frac{1}{2}\left(x + \frac{1}{2}\right) + \frac{1}{4};$$

if the number of terms is even, the sum will be $= -\frac{1}{2}x$ and for an odd number of terms $= +\frac{1}{2}x + \frac{1}{2}$.

$$\text{III.}\quad 1 - 2^2 + 3^2 - 4^2 + \cdots \mp x^2 = \mp\frac{1}{2}\left(x^2 + x\right)$$

for an even number $= -\frac{1}{2}xx - \frac{1}{2}x$ and for an odd number $= +\frac{1}{2}xx + \frac{1}{2}x$.

$$\text{IV.}\quad 1 - 2^3 + 3^3 - 4^3 + \cdots \mp x^3 = \mp\frac{1}{2}\left(x^3 + \frac{3}{2}xx - \frac{1}{4}\right) - \frac{1}{8};$$

for even number of terms $= -\frac{1}{2}x^3 - \frac{3}{4}x^2$ and for an odd number $= \frac{1}{2}x^3 + \frac{3}{4}x^2 - \frac{1}{4}$.

$$\text{V.}\quad 1 - 2^4 + 3^4 - 4^4 + \cdots \mp x^4 = \mp\frac{1}{2}(x^4 - 2x^3 - x);$$

for an even number of terms $= -\frac{1}{2}x^4 - x^3 + \frac{1}{2}x^2$ and for an odd number $= \frac{1}{2}x^4 + x^3 - \frac{1}{2}x$ etc.

**§185** Therefore, it is clear that in the even powers except $n = 0$ the constant to be added vanishes and in these cases the sum of an even number or an odd number of terms only differ in regard of the sign. If therefore $x$ was an infinite number, since it is neither even nor odd, this consideration is not valid and the ambiguous signs are to be rejected; hence, it follows that the sum of series continued to to infinity of this kind are expressed by means of the constant to be added alone.

Therefore, it will be

$$\begin{aligned}
&1-1\ +1\ -1\ +\text{etc. to infinity} &&= +\frac{1}{2}\\
&1-2\ +3\ -4\ +\text{etc.} &&= +\frac{A}{4} = +\frac{(2^2-1)\mathfrak{A}}{2}\\
&1-2^2+3^2-4^2+\text{etc.} &&= 0\\
&1-2^3+3^3-4^3+\text{etc.} &&= -\frac{B}{8} = -\frac{(2^4-1)\mathfrak{B}}{4}\\
&1-2^4+3^4-4^4+\text{etc.} &&= 0\\
&1-2^5+3^5-4^5+\text{etc.} &&= +\frac{C}{12} = +\frac{(2^6-1)\mathfrak{C}}{6}\\
&1-2^6+3^6-4^6+\text{etc.} &&= 0\\
&1-2^7+3^7-4^7+\text{etc.} &&= -\frac{D}{16} = -\frac{(2^8-1)\mathfrak{D}}{8}\\
&\qquad\qquad\text{etc.}
\end{aligned}$$

These same sums are found by means of the method treated above [§ 8] to sum series in which the signs $+$ and $-$ alternate.

**§186** If for $n$ one sets negative numbers, the expression of the sums runs to infinite. Let $n = -1$; the sum of the series will be

$$\begin{aligned}
&1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots\mp\frac{1}{x}\\
&= \mp\frac{1}{2}\left(\frac{1}{x}-\frac{A}{2x^2}+\frac{B}{4x^4}-\frac{C}{6x^6}+\frac{D}{8x^8}-\text{etc.}\right)+\text{Const.}
\end{aligned}$$

Because here the constant cannot be defined from the case $x = 0$, it is to be defined from another case. Put $x = 1$ and because of the sum $= 1$ and the inferior sign holds it will be

$$\text{Const.} = 1 - \frac{1}{2}\left(\frac{1}{1} - \frac{A}{2} + \frac{B}{4} - \frac{C}{6} + \text{etc.}\right)$$

or

$$\text{Const.} = \frac{1}{2} + \frac{A}{4} - \frac{B}{8} + \frac{C}{12} - \frac{D}{16} + \text{etc.}$$

Or put $x = 2$; because of the sum $= \frac{1}{2}$ and the superior sign it will be found

$$\text{Const.} = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} - \frac{A}{2 \cdot 2^2} + \frac{B}{4 \cdot 2^4} - \frac{C}{6 \cdot 2^6} + \text{etc.}\right)$$

or

$$\text{Const.} = \frac{3}{4} - \frac{A}{4 \cdot 2^2} + \frac{B}{8 \cdot 2^4} - \frac{C}{12 \cdot 2^6} + \frac{D}{16 \cdot 2^8} - \text{etc.}$$

But if one puts $x = 4$, it will be

$$\text{Const.} = \frac{17}{24} - \frac{A}{4 \cdot 4^2} + \frac{B}{8 \cdot 4^4} - \frac{C}{12 \cdot 4^6} + \frac{D}{16 \cdot 4^8} - \text{etc.}$$

But no matter how the constant is defined, the same value will arise, which at the same time will indicate the sum of the series continued to infinity which is $= \ln 2$.

**§187** Additionally, from these new numbers $A$, $B$, $C$, $D$, $E$ etc. the sums of the series of the reciprocal powers in which only the odd numbers occur can be summed in a convenient way. For, if one puts

$$1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \frac{1}{6^{2n}} + \text{etc.} = s$$

it will be

$$+\frac{1}{2^{2n}} \qquad + \frac{1}{4^{2n}} \qquad + \frac{1}{6^{2n}} + \text{etc.} = \frac{s}{2^{2n}},$$

which subtracted from the latter leaves over

$$1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \text{etc.} = \frac{(2^{2n} - 1)s}{2^{2n}},$$

Because we exhibited the values of $s$ for the single numbers $n$ already above (§ 125), it will be

$$\begin{aligned}
1++\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\text{etc.} &= \frac{A}{1\cdot 2}\cdot\frac{\pi^2}{4}\\
1++\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\text{etc.} &= \frac{B}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{\pi^4}{4}\\
1++\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\text{etc.} &= \frac{C}{1\cdot 2\cdot 3\cdots 6}\cdot\frac{\pi^6}{4}\\
1++\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\text{etc.} &= \frac{D}{1\cdot 2\cdot 3\cdots 8}\cdot\frac{\pi^8}{4}\\
1++\frac{1}{3^{10}}+\frac{1}{5^{10}}+\frac{1}{7^{10}}+\text{etc.} &= \frac{E}{1\cdot 2\cdot 3\cdots 10}\cdot\frac{\pi^2}{4}\\
&\text{etc.}
\end{aligned}$$

But if all numbers go in and the signs alternate, since it will be

$$1-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\frac{1}{4^{2n}}+\text{etc.}=\frac{(2^{2n}-1)s-s}{2^{2n}},$$

one will have

$$\begin{aligned}
1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\text{etc.} &= \frac{A-2\mathfrak{A}}{1\cdot 2}\cdot\frac{\pi^2}{4}=\frac{(2-1)\mathfrak{A}}{1\cdot 2}\cdot\pi^2\\
1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\frac{1}{5^4}-\text{etc.} &= \frac{B-2\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{\pi^2}{4}=\frac{(2^3-1)\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}\cdot\pi^4\\
1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\frac{1}{5^6}-\text{etc.} &= \frac{C-2\mathfrak{C}}{1\cdot 2\cdots 6}\cdot\frac{\pi^2}{4}=\frac{(2^5-1)\mathfrak{C}}{1\cdot 2\cdots 6}\cdot\pi^6\\
1-\frac{1}{2^8}+\frac{1}{3^8}-\frac{1}{4^8}+\frac{1}{5^8}-\text{etc.} &= \frac{D-2\mathfrak{D}}{1\cdot 2\cdots 8}\cdot\frac{\pi^2}{4}=\frac{(2^7-1)\mathfrak{D}}{1\cdot 2\cdots 8}\cdot\pi^2\\
1-\frac{1}{2^{10}}+\frac{1}{3^{10}}-\frac{1}{4^{10}}+\frac{1}{5^{10}}-\text{etc.} &= \frac{E-2\mathfrak{E}}{1\cdot 2\cdots 10}\cdot\frac{\pi^2}{4}=\frac{(2^9-1)\mathfrak{E}}{1\cdot 2\cdots 10}\cdot\pi^2\\
&\text{etc.}
\end{aligned}$$

**§188** As we up to now contemplated a series whose terms are the products of terms from the geometric progression $p$, $p^2$, $p^3$ etc. and terms of the arbitrary series $a$, $b$, $c$ etc., so we will be able to prosecute the series whose general terms are the products of terms of two arbitrary series of which the one is assumed to be known. Let this series be known

$$\begin{array}{ccccc} 1 & 2 & 3 & & z \\ A + & B + & C & + \cdots + & Z, \end{array}$$

the other on the other hand unknown

$$a + b + c + \cdots + z$$

and let the sum of this series be in question

$$Aa + Bb + Cc + \cdots + Zz,$$

which shall be put $= Zs$. In the known series let the penultimate term be $= Y$ and having put $x - 1$ instead of $x$ the expression of the sum $S.Zz$ will go over into

$$Y\left(s - \frac{ds}{dx} + \frac{dds}{2dx^2} - \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} - \text{etc.}\right).$$

Since this sum expresses the series $Zs$ diminished by the last term $Zz$, it will be

$$Zs - Zz = Ys - \frac{Yds}{dx} + \frac{Ydds}{2dx^2} - \frac{Yd^3s}{6dx^3} + \text{etc.},$$

which equation contains the relation how the sum $Zs$ depends on $Y$, $Z$ and $z$.

**§189** To resolve this equation at first neglect the differential terms and it will be

$$s = \frac{Zz}{Z - Y};$$

put this value $\frac{Zz}{Z-Y} = P^{\mathrm{I}}$ and let the true sum be $s = P^{\mathrm{I}} + p$; having substituted this value in the equation it will be

$$\begin{aligned}(Z - Y)p \ = \ & -\frac{YdP^{\mathrm{I}}}{dx} + \frac{YddP^{\mathrm{I}}}{2dx^2} - \text{etc.} \\ & -\frac{Ydp}{dx} + \frac{Yddp}{2dx^2} - \text{etc.};\end{aligned}$$

add $YP^{\mathrm{I}}$ on both sides, and since $P^{\mathrm{I}} - \frac{dP^{\mathrm{I}}}{dx} + \frac{YddP^{\mathrm{I}}}{2dx^2}$ $-$etc. is the value of $P^{\mathrm{I}}$ which arises, if instead of $x$ one puts $x-1$, let this value be $P$ and it will be

$$(Z-Y)p + YP^{\mathrm{I}} = YP - \frac{Ydp}{dx} + \frac{Yddp}{2dx^2} - \text{etc.},$$

whence neglecting the differentials it will be

$$p = \frac{Y(P-P^{\mathrm{I}})}{Z-Y}.$$

Put $\frac{Y(P-P^{\mathrm{I}})}{Z-Y} = Q^{\mathrm{I}}$ and let $p = Q^{\mathrm{I}} + q$; it will be

$$(Z-Y)q = -\frac{Y(dQ^{\mathrm{I}}+dq)}{dx} + \frac{Y(ddQ^{\mathrm{I}}+ddq)}{2dx^2} - \text{etc.}$$

and having put $Q$ for the value of $Q^{\mathrm{I}}$ which it takes, if instead of $x$ one writes $x-1$, it will be

$$(Z-Y)q + YQ^{\mathrm{I}} = YQ - \frac{Ydq}{dx} + \frac{Yddq}{2dx^2} - \text{etc.},$$

whence neglecting the differentials it is

$$q = \frac{Y(Q-Q^{\mathrm{I}})}{Z-Y}.$$

Put $\frac{Y(Q-Q^{\mathrm{I}})}{Z-Y} = R^{\mathrm{I}}$ and let the true value be $q = R^{\mathrm{I}} + r$ and in similar way one will find

$$r = \frac{Y(R-R^{\mathrm{I}})}{Z-Y};$$

and by proceeding this way the sum in question will be

$$Zs = Z(P^{\mathrm{I}} + Q^{\mathrm{I}} + R^{\mathrm{I}} + \text{etc.}).$$

**§190** Therefore, having propounded any series

$$Aa + Bb + Cc + \cdots + Yy + Zz$$

its sum will be defined the following way

| Put | | having put $x-1$ instead of $x$ |
|---|---|---|
| $\dfrac{Zz}{Z-Y}$ | $= P^{\mathrm{I}}$ | and let $P^{\mathrm{I}}$ go over into $P$ |
| $\dfrac{Y(P-P^{\mathrm{I}})}{Z-Y}$ | $= Q^{\mathrm{I}}$ | and let $Q^{\mathrm{I}}$ go over into $Q$ |
| $\dfrac{Y(Q-Q^{\mathrm{I}})}{Z-Y}$ | $= R^{\mathrm{I}}$ | and let $R^{\mathrm{I}}$ go over into $R$ |
| $\dfrac{Y(R-R^{\mathrm{I}})}{Z-Y}$ | $= S^{\mathrm{I}}$ | and let $S^{\mathrm{I}}$ go over into $S$ |
| | etc. | |

Having found these values the sum of the series will be

$$= ZP^{\mathrm{I}} + ZQ^{\mathrm{I}} + ZR^{\mathrm{I}} + ZS^{\mathrm{I}} + \text{etc.}$$

$+$ Constant which renders the sum $= 0$, if one puts $x = 0$, or what reduces to the same which causes that it satisfies a certain case.

**§191** This formula, since it does not contain any differentials, is in the most cases applied most easily and even often will exhibit the true sum. So if this series is propounded

$$p + 4p^2 + 9p^3 + 16p^4 + \cdots + x^2p^x,$$

let $Z = p^x$ and $z = x^2$; it will be $Y = p^{x-1}$ and $\frac{Z}{Z-Y} = \frac{p}{p-1}$ and $\frac{Y}{Z-Y} = \frac{1}{p-1}$. Hence, it will become

$$P^{\mathrm{I}} = \frac{px^2}{p-1} \qquad P = \frac{pxx - 2px + p}{p-1}$$

$$Q^{\mathrm{I}} = \frac{-2px + p}{(p-1)^2} \qquad Q = \frac{-2px + 3p}{(p-1)^2}$$

$$R^{\mathrm{I}} = \frac{2p}{(p-1)^3} \qquad R = \frac{2p}{(p-1)^3}$$

$$S^{\mathrm{I}} = 0$$

and all remaining vanish; hence, the sum will be

$$= p^x\left(\frac{px^2}{p-1} - \frac{2px-p}{(p-1)^2} + \frac{2p}{(p-1)^3}\right) - \frac{p}{(p-1)^2} - \frac{2p}{(p-1)^3}$$
$$= p^{x+1}\left(\frac{x^2}{p-1} - \frac{2x}{(p-1)^2} + \frac{p+1}{(p-1)^3}\right) - \frac{p(p+1)}{(p-1)^3}.$$

as we already found above [§ 176].

**§192** In the same way we got to this expression of the sum we will be able to find another expression, if the propounded series is not composed of two others; this can mainly be used in those cases, in which in the preceding expression one gets to vanishing denominators. Therefore, let this series be propounded

$$\begin{aligned} &\quad\ 1 \quad\ 2 \quad\ 3 \quad\ 4 \qquad\qquad\quad x \\ s &= a + b + c + d\ +\ \cdots\ + z; \end{aligned}$$

since having put $x-1$ instead of $x$ the sum is truncated by the last term, it will be

$$s - z = s - \frac{ds}{dx} + \frac{dds}{2dx^2} - \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} - \text{etc.}$$

or

$$z = \frac{ds}{dx} - \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} - \frac{d^4s}{24dx^4} + \text{etc.}$$

Since here the sum $s$ does not occur, neglect the higher differentials and it will be $s = \int zdx$; put $\int zdx = P^{\mathrm{I}}$ whose value shall go over into $P$, if for $x$ one writes $x-1$, and let the true value be $P^{\mathrm{I}}$; it will be

$$z = \frac{dP^{\mathrm{I}}}{dx} - \frac{ddP^{\mathrm{I}}}{2dx^2} + \text{etc.} + \frac{dp}{dx} - \frac{ddp}{2dx^2} + \text{etc.};$$

since it is

$$P = P^{\mathrm{I}} - \frac{dP^{\mathrm{I}}}{dx} + \frac{ddP^{\mathrm{I}}}{2dx^2} - \text{etc.},$$

it will be

$$z - P^{\mathrm{I}} + P = \frac{dp}{dx} - \frac{ddp}{2dx^2} + \text{etc.},$$

whence it is

$$p = \int (z - P^{\mathrm{I}} - P)dx.$$

If one further puts $\int (z - P^{\mathrm{I}} + P)dx = Q^{\mathrm{I}}$ and this value shall go over into $Q$ having put $x-1$ instead of $x$, let

$$\int (z - P^{\mathrm{I}} + P - Q^{\mathrm{I}} + Q)dx = R^{\mathrm{I}} = Q^{\mathrm{I}} - \int (Q^{\mathrm{I}} - Q)dx,$$

further

$$R^{\mathrm{I}} - \int (R^{\mathrm{I}} - R)dx = S^{\mathrm{I}}$$

etc.; the sum in question will be

$$s = P^{\mathrm{I}} + Q^{\mathrm{I}} + R^{\mathrm{I}} + S^{\mathrm{I}} + \text{etc.} + \text{Const.},$$

by means of which one single case shall be satisfied.

**§193** Having changed the letters a little bit, this summation reduced to this. Having propounded this series to be summed

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & & & x \\ s = & a + & b + & c + & d & + & \cdots & + z \end{array}$$

| put | | having put | $x-1$ | instead of | $x$ |
|---|---|---|---|---|---|
| $\int zdx$ | $= P$ | and let | $P$ | go over into | $p$ |
| $P - \int (P-p)dx$ | $= Q$ | and let | $Q$ | go over into | $q$ |
| $Q - \int (Q-q)dx$ | $= R$ | and let | $R$ | go over into | $r$ |
| etc. | | | | | |

having found which values the sum in question will be

$$s = P + Q + R + S + \text{etc.}$$

this expression will immediately show the sum, if these integral formulas can be exhibited. Let, to show its use by means of an example, $z = xx + x$ and it will be

$$P = \frac{1}{3}x^3 + \frac{1}{2}xx, \quad p = \frac{1}{3}x^3 - \frac{1}{2}xx + \frac{1}{6}, \quad P - p = xx - \frac{1}{6}$$

and

$$\int (P-p)dx = \frac{1}{3}x^3 - \frac{1}{6}x;$$

$$Q = \frac{1}{2}xx + \frac{1}{6}x, \quad q = \frac{1}{2}xx - \frac{5}{6}x + \frac{1}{3}, \quad Q - q = x - \frac{1}{3}$$

and

$$\int (Q-q)dx = \frac{1}{2}xx - \frac{1}{3}x;$$

$$R = \frac{1}{2}x, \quad r = \frac{1}{2}x - \frac{1}{2}, \quad R - r = \frac{1}{2}$$

and

$$\int (R-r)dx = \frac{1}{2}x;$$

$S = 0$ and the remaining values vanish. Hence, the sum in question will be

$$\left.\begin{aligned} &\tfrac{1}{3}x^3 + \tfrac{1}{2}xx \\ &\qquad + \tfrac{1}{2}xx + \tfrac{1}{6}x \\ &\qquad\qquad + \tfrac{1}{2}x \end{aligned}\right\} = \frac{1}{3}x^3 + xx + \frac{2}{3}x = \frac{1}{3}x(x+1)(x+2).$$

And this way the sum of all series whose general terms are polynomial functions of $x$ can be found by means of continued integrations. From these it is easily seen what a large field the doctrine of the summation of series occupies and several volumes will not suffice to capture all methods which one has and which can still be thought of.

**§194** Up to now we investigated sums of series from the first term up to the one whose index is $x$ having known which by putting $x = \infty$ the sum of the series continued to infinity also becomes known. But often this is more easily done, if not the sum of terms from the first to that one whose index is $x$ but the sum of the terms from the one whose index is $x$ up to infinity is sought after, and in this case especially the last expressions become more tractable. Therefore, let a series be propounded whose general term or the one corresponding to the index $x$ shall be $= z$, the following corresponding to the index $x+1$ shall be $= z^{\mathrm{I}}$ and the ones following after this one $z^{\mathrm{II}}$, $z^{\mathrm{III}}$ etc. and let the sum of this infinite series be in question

$$\begin{array}{ccccccccc} x & & x+1 & & x+2 & & x+3 & & \text{etc.} \\ s = z & + & z^{\mathrm{I}} & + & z^{\mathrm{II}} & + & z^{\mathrm{III}} & + & \text{etc. to infinity} \end{array}$$

Therefore, the sum $s$ will be a function of $x$; if in this one one puts $x+1$ instead of $x$, the first sum truncated by the term $z$ will arise. Since by means of this change $s$ goes over into

$$s + \frac{ds}{dx} + \frac{dds}{2dx^2} + \text{etc.},$$

it will be

$$s - z = s + \frac{ds}{dx} + \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} + \frac{d^5s}{120dx^5} + \text{etc.}$$

or

$$0 = z + \frac{ds}{dx} + \frac{dds}{2dx^2} + \frac{d^3s}{6dx^3} + \frac{d^4s}{24dx^4} + \frac{d^5s}{120dx^5} + \text{etc.}$$

**§195** If we now argue as before, having neglected the higher differentials it will be $s = C - \int zdx$. Therefore, put $\int zdx = P$ and let the true value be $s = C - P + p$; it will be

$$\begin{aligned} 0 = z - \frac{dP}{dx} - \frac{ddP}{2dx^2} - \frac{d^3P}{6dx^3} - \text{etc.} \\ + \frac{dp}{dx} + \frac{ddp}{2dx^2} + \frac{d^3p}{6dx^3} + \text{etc.} \end{aligned}$$

Let $P$ go over into $P^{\mathrm{I}}$, if instead of $x$ one puts $x+1$, and it will be

$$0 = z + P - P^{\mathrm{I}} + \frac{dp}{dx} + \frac{ddp}{2dx^2} + \frac{d^3p}{6dx^3} + \text{etc.}$$

Hence, having neglected the higher differentials it will be $p = \int (P^{\mathrm{I}} - P)dx - P$. Set $\int (P^{\mathrm{I}} - P)dx - P = -Q$ and let $p = -Q + q$; it will be

$$0 = z + P - P^{\mathrm{I}} - \frac{dQ}{dx} - \frac{ddQ}{2dx^2} - \text{etc.} + \frac{dq}{dx} + \frac{ddq}{2dx^2} + \text{etc.}$$

Let $Q$ go over into $Q^{\mathrm{I}}$, if instead of $x$ one puts $x + 1$, and it will be

$$0 = z + P - P^{\mathrm{I}} + Q - Q^{\mathrm{I}} + \frac{dq}{dx} + \frac{ddq}{2dx^2} + \text{etc.},$$

whence $q = \int (Q^{\mathrm{I}} - Q)dx - Q$ follows. Therefore, if the sign I attached to the quantity denotes its value which it takes having put $x + 1$ instead of $x$, and one puts

$$\begin{aligned} \int zdx &= P \\ P - \int (P^{\mathrm{I}} - P)dx &= Q \\ Q - \int (Q^{\mathrm{I}} - Q)dx &= R \\ R - \int (R^{\mathrm{I}} - R)dx &= S \\ \text{etc.,} & \end{aligned}$$

the sum of the propounded series $z + z^{\mathrm{I}} + z^{\mathrm{II}} + z^{\mathrm{III}} + z^{\mathrm{IV}} + \text{etc.}$ will be

$$= C - P - Q - R - S - \text{etc.},$$

where the constant $C$ has to defined in such a way that having put $x = \infty$ the total sum vanishes. But since the application of this expression requires integrations, it is not possible to show its use here.

**§196** But to avoid integral formulas let us set the sum of the series $= ys$ while $y$ is any known function of $x$ whose values $y^{\mathrm{I}}$, $y^{\mathrm{II}}$ etc. which arise by putting $x + 1$, $x + 2$ etc. instead of $x$ will be known. If one now puts $x + 1$ instead of $x$, the superior series truncated by the first term will arise, whose sum will therefore be

$$y^{\mathrm{I}}\left(s+\frac{ds}{dx}+\frac{dds}{2dx^2}+\frac{d^3s}{6dx^3}+\text{etc.}\right)=ys-z$$

or

$$z+\frac{y^{\mathrm{I}}ds}{dx}+\frac{y^{\mathrm{I}}dds}{2dx^2}+\frac{y^{\mathrm{I}}d^3s}{6dx^3}+\text{etc.}=(y-y^{\mathrm{I}})s,$$

whence having neglected the differentials $s=\frac{z}{y-y^{\mathrm{I}}}$ arises. Set $\frac{z}{y^{\mathrm{I}}-y}=P$ and let the true value be $s=-P+p$; it will be

$$\left.\begin{array}{l}-\frac{y^{\mathrm{I}}dP}{dx}-\frac{y^{\mathrm{I}}ddP}{2dx^2}-\frac{y^{\mathrm{I}}d^3P}{6dx^3}-\text{etc.}\\ +\frac{y^{\mathrm{I}}dp}{dx}+\frac{y^{\mathrm{I}}ddp}{2dx^2}+\frac{y^{\mathrm{I}}d^3p}{6dx^3}-\text{etc.}\end{array}\right\}=(y-y^{\mathrm{I}})p$$

and hence

$$\frac{y^{\mathrm{I}}dp}{dx}+\frac{y^{\mathrm{I}}ddp}{2dx^2}+\frac{y^{\mathrm{I}}d^3p}{6dx^3}+\text{etc.}=y^{\mathrm{I}}(P^{\mathrm{I}}-P)-(y^{\mathrm{I}}-y)p.$$

Set $Q=\frac{y^{\mathrm{I}}(P^{\mathrm{I}}-P)}{y^{\mathrm{I}}-y}$ and let $p=Q+q$; it will be

$$y^{\mathrm{I}}(Q^{\mathrm{I}}-Q)+y^{\mathrm{I}}\left(\frac{dq}{dx}+\frac{ddq}{2dx^2}+\text{etc.}\right)=-(y^{\mathrm{I}}-y)q.$$

Put $R=\frac{y^{\mathrm{I}}(Q^{\mathrm{I}}-Q)}{y^{\mathrm{I}}-y}$ and let be $q=-R+r$.

And if we proceed this way, the sum of the propounded series

$$z+z^{\mathrm{I}}+z^{\mathrm{II}}+z^{\mathrm{III}}+z^{\mathrm{IV}}+\text{etc.}$$

will be found the following way. Having taken a function of $x$ ad libitum which function shall be $=y$, set

$$\begin{aligned}P&=\frac{z}{y^{\mathrm{I}}-y}=\frac{z}{\Delta y}\\ Q&=\frac{y^{\mathrm{I}}(P^{\mathrm{I}}-P)}{y^{\mathrm{I}}-y}=\frac{y\Delta P}{\Delta y}+\Delta P\end{aligned}$$

$$R = \frac{y^{\mathrm{I}}(Q^{\mathrm{I}} - Q)}{y^{\mathrm{I}} - y} = \frac{y\Delta Q}{\Delta y} + \Delta Q$$

$$S = \frac{y^{\mathrm{I}}(R^{\mathrm{I}} - R)}{y^{\mathrm{I}} - y} = \frac{y\Delta R}{\Delta y} + \Delta R$$

etc.

And hence the sum in question will be

$$= C - Py + Qy - Ry + Sy - \text{etc.}$$

having taken a constant of such a kind for $C$ that having put $x = \infty$ the sum vanishes.

**§197** Assume $y = a^x$; because of $y^{\mathrm{I}} = a^{x+1}$ it will be $y^{\mathrm{I}} - y = a^x(a-1)$, whence it will become

$$P = \frac{z}{a^x(a-1)} \qquad P^{\mathrm{I}} = \frac{z^{\mathrm{I}}}{a^{x+1}(a-1)}$$

$$Q = \frac{a(P^{\mathrm{I}} - P)}{a-1} = \frac{z^{\mathrm{I}} - az}{a^x(a-1)^2} \qquad Q^{\mathrm{I}} = \frac{z^{\mathrm{II}} - az^{\mathrm{I}}}{a^{x+1}(a-1)^2}$$

$$R = \frac{a(Q^{\mathrm{I}} - Q)}{a-1} = \frac{z^{\mathrm{II}} - 2az^{\mathrm{I}} + aaz}{a^x(a-1)^3} \qquad R^{\mathrm{I}} = \frac{z^{\mathrm{III}} - 2az^{\mathrm{II}} + aaz^{\mathrm{I}}}{a^{x+1}(a-1)^2}$$

$$S = \frac{a(R^{\mathrm{I}} - R)}{a-1} = \frac{z^{\mathrm{III}} - 3az^{\mathrm{II}} + 3a^2z^{\mathrm{I}} - a^3}{a^x(a-1)^4}$$

etc.

Therefore, the sum of the propounded series will be

$$C - \frac{z}{a-1} + \frac{z^{\mathrm{I}} - az}{(a-1)^2} - \frac{z^{\mathrm{II}} - 2az^{\mathrm{I}} + a^2z}{(a-1)^3} + \frac{z^{\mathrm{III}} - 3az^{\mathrm{II}} + 3a^2z^{\mathrm{I}} - a^3z}{(a-1)^4} - \text{etc.}$$

This same expression of the sum was indeed found already above in the first chapter. Hence, by assuming other values for $y$ infinitely other expressions can be found, whence the one, which is accommodated to each case the most, can be elected.